# On Suspended Particles in Tube Flow

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**The paper considers the effect on fluid flow in a tube when rotationally symmetric particles follow the flow on the tube axis. It is found that a region of disturbed flow, similar to inlet flow, adheres to the particle. A model describing this disturbed flow is suggested, and an expression for the change in the apparent viscosity of the flow is developed. It is found that the accompanying disturbance may have an appreciable effect on the flow. The model should be applicable when the tube Reynolds number is larger than 10.**

Depending on flow conditions and fluid viscosity, the presence of a suspended particle in the steady flow of a fluid through a tube may be felt some distance from the particle. In the case of a nonrotating particle the disturbance of the flow must be approximately the same as if a part of a tube section, of the same shape as the particle cross-sectional area, is held at a constant uniform velocity. This latter problem has been studied theoretically (*1* to *5*) and experimentally (*6*) in the special case of fluid entering a tube at a cross section by uniform velocity.

In such inlet flow a region of nonuniform velocity grows inward from the walls, until the flow is wholly parabolic. In Schiller's (*2*) approximation, the velocity profile in the transition length consists of a central core of cross-sectionally uniform velocity and an outer region with parabolic velocity distribution, the apex of the parabola meeting the uniform part (Figure 1*a*). At the particle section, on the other hand, the velocity distribution is given by a uniform central region, covered by the particle, and, considering axially symmetric particles, an outer paraboloid region with apex at the tube axis (Figure 1*b*). In this case the decay of the uniform region must start at the particle surface. In addition to the tube Reynolds number involved in the solutions to the inlet flow, the particle Reynolds number then enters the solutions. For a given tube Reynolds number the solutions must therefore be developed separately for each particle size. It is thus difficult to extend the models for the inlet flow to the case of particle flow.

However, the handling of the problem is simplified if the outer parabolic part of Schiller's model is replaced by a truncated paraboloid, with its apex at the tube axis (Figure 1*c*). The basic surface defining the boundary between paraboloid and uniform velocity is then the same for all particles and can be determined once and for all by the appropriately modified inlet flow.

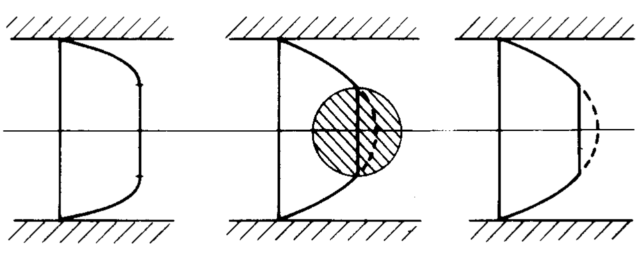


Fig. 1. (a) Schiller's model for the velocity profile in inlet flow. (b) Velocity profile at a particle in tube flow. (c) Present model for the velocity profile in inlet and particle flow.

## APPROXIMATE MODEL OF THE DISTURBANCE

If $a$ is the radius of the uniform central core and $r$ the distance from the tube axis, both dimensionless as proportions of the tube radius $R$, the modified velocity profile in any section is

$$u = U \qquad r < a$$

$$u = U\frac{1-r^2}{1-a^2} \qquad a \leq r < 1$$

If the fluid is incompressible, the flux is constant along the tube with an average velocity $U_0$, such that

$$R^2\int_0^1 2\pi U_0 r\,dr = R^2\int_0^a 2\pi U r\,dr + R^2\int_a^1 2\pi U\frac{1-r^2}{1-a^2}r\,dr$$

and

$$U_0 = \frac{1}{2}U(1+a^2)$$

or, with $U/U_0 = q$

$$q(1+a^2) = 2 \tag{1}$$

At the inlet $a = 1$ and at the end of the influence region $a = 0$, giving $q = 1$ and $q = 2$, respectively; that is, the velocity over the section at the inlet is uniform and the settled flow is parabolic with appropriate axial velocity. Specifically, if the radius of the particle is $a_0$, its velocity will be given by the exact relation

$$q = \frac{2}{1+a_0^2} \tag{2}$$

Following Schiller, it is assumed that the pressure is a function of the distance $x = Rs$ from the inlet along the tube axis alone. It is further assumed that the friction term $\mu\frac{\partial^2 U}{\partial x^2}$ can be neglected in the core. The pressure in the core is then given by Bernoulli's equation, and the pressure force acting on a short length $\Delta x = R\Delta s$ of tube in the influence region is therefore

$$R^2\int_0^1 2\pi\left(-\frac{\partial p}{\partial x}\right)\Delta x\,r\,dr = \pi R^2 U_0^2 \rho q\frac{\partial q}{\partial s}\Delta s \tag{3}$$

The friction force acting on the wall of the same length of tube is

$$-2\pi R\mu\left|\frac{1}{R}\frac{\partial u}{\partial r}\right|_{r=1}\Delta x = -4\pi\mu R|U_0|\frac{q}{1-a^2}\Delta s \tag{4}$$

The change of momentum of the fluid in the tube length

$$2\pi R^2 \rho \Delta x \int_0^1 u \frac{\partial u}{\partial x} \Delta x\, r\, dr = 2\pi R^2 \rho \Delta x U \left\{ \frac{\partial U}{\partial x} \int_0^a r\, dr \right.$$

$$\left. + \int_a^1 \frac{1-r^2}{1-a^2} \frac{\partial}{\partial x}\left( U \frac{1-r^2}{1-a^2} \right) r\, dr \right\}$$

$$= 2\pi \rho U_0^2 R^2 \Delta s q \left\{ \frac{1}{2} a^2 \frac{\partial q}{\partial s} + \right.$$

$$\left. + \frac{1}{6}(1-a^2)^2 \frac{\partial}{\partial s}\left( \frac{q}{1-a^2} \right) \right\} \quad (5)$$

is due to the pressure and friction forces, and Equations (3), (4), and (5) then give the equation

$$(1-a^2)\frac{\partial q}{\partial s} - \frac{(1-a^2)^2}{3} \frac{\partial}{\partial s}\left( \frac{q}{1-a^2} \right) = 4 \frac{1}{1-a^2} N_{Re}^{-1}$$

When $q$ is substituted from (1), this can be written as

$$\frac{(1-a^2)\left(1-\frac{a^2}{3}\right)}{(1+a^2)^2} a \frac{\partial a}{\partial s} = -N_{Re}^{-1} \quad (6)$$

and in implicit form the equation for the basic core surface is

$$\ln \frac{1+a^2}{2} - \frac{1}{6} \frac{a^4 + 4a^2 - 5}{1+a^2} = s N_{Re}^{-1} \quad (7)$$

since $a = 1$ corresponds to $s = 0$.

Equations (1) and (6) give the relation

$$\frac{2}{3}(q-1) - \ln q + \frac{1}{3} \frac{q-1}{q} = s N_{Re}^{-1}$$

between $q$ and $sN_{Re}^{-1}$. This relation has been plotted in Figure 2 together with Nikuradses' (6) observations and values from Schiller's model. Although the core apex, where $q = 2$, is found at

$$s^* = 0.14 N_{Re}$$

in fair agreement with observations, the present approximation to the inlet flow is clearly poorer than Schiller's approximation.

Equations (3), (4), and (5) are independent of the direction of flow, or, equivalently, whether $s$ is measured upstream or downstream. Considering a piston driving a tube flow, the core surface is then the same whether the piston is pushed or pulled. This is of course independent of the present modification to the velocity distribution. Similarly any axially symmetric particle moving coaxially in the tube must have an accompanying longitudinally symmetric core.

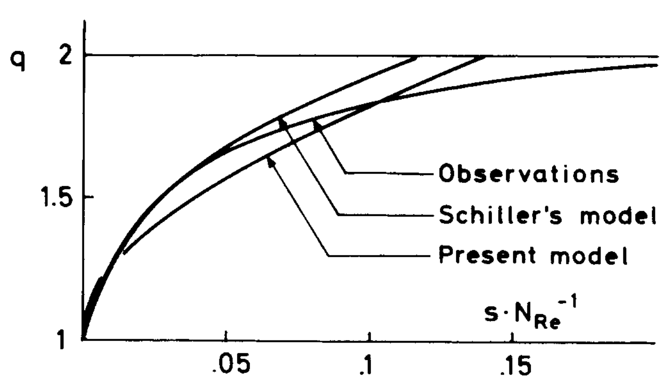


Fig. 2. Observed and theoretical values for the velocity distribution on the axis in inlet flow.

With the present model an approximation to this core is found by fitting the particle into the basic core surface, shown in Figure 3*a* as computed from Equation (7). A crude picture of the cross-sectionally uniform velocity region of a particle may be constructed as follows. The properly scaled particle is placed inside the basic core surface and pushed toward one apex until it touches the surface. The surface between the circle of contact and apex is hitched to the particle, and the process is repeated in the other direction. After rescaling, the assembly then gives the boundary of the core (Figure 3*b*). If the particle has the shape of a circular cylinder, the fluid cores are then fitted to the end surfaces, and the possible resistance in the influence region would add to the resistance along the cylinder. A spherical particle, on the other hand, might be completely enclosed at large Reynolds numbers. As the Reynolds number is lowered the cores get shorter, an equatorial region of sphere surface emerges, and finally the cores disappear "inside" the sphere. But then the model breaks down. Generally, considering particles without flat ends, the dimensionless length of the particle must be less than the length of the fluid cores, which again is less than the length of the basic core.

It may also be mentioned that according to the model the velocity is uniform over the cross sections in the core, and any such section should thus be either all fluid or all particle. This will not be even approximately correct for ellipsoid and spherical particles at the lowest Reynolds numbers.

Furthermore, independent of particle shape, the present model is based on the assumption that the viscous forces are negligible in the core. Since the inertial forces are of the order of

$$\rho u \frac{\partial u}{\partial x} \sim \rho U_0 \frac{U_0}{Rs^*} \sim \frac{\rho U_0^2}{R \cdot .14 \cdot N_{Re}}$$

while the viscous forces are of the order of

$$\mu \frac{\partial^2 u}{\partial x^2} \sim \frac{\mu U_0}{(Rs^*)^2} \sim \frac{\mu U_0}{(R \cdot .14 \cdot N_{Re})^2}$$

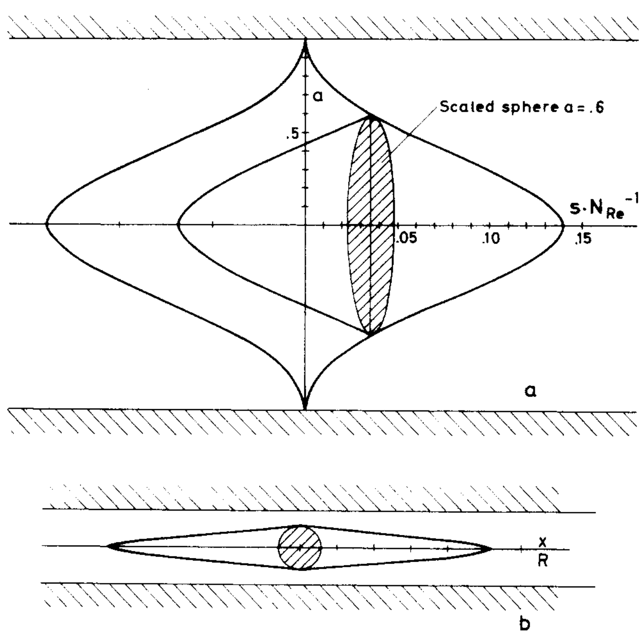


Fig. 3. (*a*) The basic core, the boundary between uniform and nonuniform cross-sectional velocity, as computed from Equation (7), with the core surface of a spherical particle of radius $a_0 = 0.6$ at $N_{Re} = 50$. (*b*) The actual shape of the core surface, with the same radial and axial scales.

this implies that

$$0.14 \cdot \frac{\rho U_0 R}{\mu} \cdot N_{Re} = 0.14 \cdot (N_{Re})^2 >> 1$$

and

$$(N_{Re})^2 >> 0.14^{-1} \approx 7$$

that is, the tube Reynolds number must at least be of the order of 10

$$N_{Re} > 10$$

## EFFECT OF THE DISTURBANCE ON THE TUBE FLOW

At low Reynolds numbers and for long particles, the particle surface itself will contribute significantly to the total core boundary, and thus to the resistance induced. In other cases the main contribution stems from the fluid part of the core. The following analysis considers the influence on the tube flow of this cross-sectionally uniform velocity flow adhering to the particles.

The Poiseulle flow in the tube outside the region disturbed by the particle is everywhere the same due to the continuity of the motion. The pressure forces acting on the disturbed region must then be balanced by friction forces on the same length of tube. If the pressure difference between the ends of the disturbed region is $\delta p$, therefore

$$R^2 \int_0^1 \delta p \cdot 2\pi r dr = 2 \int_{s_0}^{s^*} 2\pi \mu R \left| \frac{1}{R} \frac{\partial u}{\partial r} \right|_{r=1} R ds \quad (8)$$

where $s_0$ denotes the section where core and particle radii are equal, $a = a_0$, and the disturbed length is $2R(s^* - s_0)$. Using Equation (4) the pressure drop associated with the particle is then

$$\delta p = \frac{8\mu U_0}{R} \int_{s_0}^{s^*} \frac{q}{1 - a^2} ds$$

$$= \frac{8\mu U_0}{R} 2 \int_{s_0}^{s^*} \frac{1}{1 - a^4} ds = \frac{8\mu U_0}{R} \cdot 2I_0^* \quad (9)$$

If a length $L$ of tube, over which the pressure drop is $\Delta p$, contains $n$ particles with noninterfering disturbed regions, the pressure drop left to drive the Poiseulle flow is $\Delta p - n\delta p$, and this flow is established in a length $L - 2nR(s^* - s_0)$ of the tube. The flux velocity is then

$$U_0 = \frac{R^2}{8\mu} \frac{\Delta p - n\delta p}{L - 2nR(s^* - s)} \quad (10)$$

If the flux is to be the same as before particles were added to the flow, then

$$U_0 = \frac{R^2}{8\mu} \frac{\Delta p_0}{L} \quad (11)$$

where $\Delta p_0$ is the original pressure drop. The relative increase in the overall pressure drop needed to compensate for the presence of particles is then, from (10) and (11)

$$\frac{\Delta p - \Delta p_0}{\Delta p_0} = \frac{n\delta p}{\Delta p_0} - 2n \frac{R}{L}(s^* - s_0) \quad (12)$$

From (9) and (11) $\delta p = 2 \frac{R}{L} I_0^* \Delta p_0$, and (12) turns into

$$\frac{\Delta p - \Delta p_0}{\Delta p} = 2n \frac{R}{L} \{I_0^* - (s^* - s_0)\} \quad (13)$$

valid as long as the disturbed regions stay separated, that is, $L \geq 2nR(s^* - s_0)$ if the particles are evenly spaced. If the particles are closer, their core surfaces will intersect at $s = s_1$, say. The distance between two particles of radius $a_0$ is then $2R(s_1 - s_0)$, and if the particles are even-spaced and $s_1$ the same for all intersections, then $L = 2nR(s_1 - s_0)$ and

$$s_1 = \frac{L}{2nR} + s_0$$

The pressure drop from particle to particle is then

$$\delta p = \frac{8\mu U_0}{R} \int_{s_0}^{s_1} \frac{q}{1 - a^2} ds$$

$$= \frac{8\mu U_0}{R} 2 \int_{s_0}^{s_1} \frac{1}{1 - a^4} ds = \frac{8\mu U_0}{R} 2I_0^1 \quad (14)$$

and substituting for $U_0$ from (11)

$$\delta p = 2 \frac{R}{L} I_0^1 \Delta p_0$$

In the particle-free flow the corresponding pressure drop is

$$\frac{\Delta p_0}{n} = \frac{\Delta p_0}{L} \cdot 2R(s_1 - s_0)$$

and the relative increase in pressure drop to keep the flow constant is then

$$\frac{\Delta p - \Delta p_0}{\Delta p_0} = 2n \frac{R}{L} I_0^1 - 1 = 2n \frac{R}{L} \{I_0^1 - (s_1 - s_0)\} \quad (15)$$

## CONCLUSIONS

If $s_1$ is put equal to $s^*$ when the cores do not intersect, the relative increase in pressure drop needed to sustain the flux after $n$ even-spaced particles of relative radius $a_0$ have been added to the tube length $L$ is, from (13) and (15)

$$\frac{\Delta p - \Delta p_0}{\Delta p_0} = 2n \frac{R}{L} \left\{ \int_{s_0}^{s_1} \frac{ds}{1 - a^4} - (s_1 - s_0) \right\}$$

$$= 2n \frac{R}{L} \int_{s_0}^{s_1} \frac{a^4 ds}{1 - a^4} \quad (16)$$

where $s_1 - s_0 = L/2nR$ when $s_1 < s^*$. Since $0 \leq a \leq 1$ the last integral is always positive, and the pressure difference must always be increased to compensate for the addition of particles to the flow.

The effect of the particles on the tube flow can alternatively be expressed as a change in the apparent viscosity of the flow. With constant flux this apparent viscosity $\mu^*$ is given by

$$U_0 = \frac{R^2}{8\mu} \frac{\Delta p_0}{L} = \frac{R^2}{8\mu^*} \frac{\Delta p}{L}$$

and substituting for $\Delta p$ from (16), this gives

$$\frac{\mu^*}{\mu} = 1 + 2n \frac{R}{L} \left\{ \int_{s_0}^{s_1} \frac{ds}{1 - a^4} - (s_1 - s_0) \right\}$$

$$= 1 + 2n \frac{R}{L} \int_{s_0}^{s_1} \frac{a^4}{1 - a^4} ds \quad (17)$$

where again the integral is non-negative and therefore $\mu^* \geq \mu$. As above $s_1 = s^*$ when $2nR(s^* - s_0) \leq L$, and $s_1 - s_0 = L/2nR$ when $s_1 < s^*$. By (15), (17) may in the latter case be written as

$$\frac{\mu^*}{\mu} = 2n \frac{R}{L} \int_{s_0}^{s_1} \frac{ds}{1 - a^4}$$

### Application of the Model to Spherical Particles

In the special case of spherical particles, where volume concentration $c$ is expressed by

$$c = \frac{\frac{4}{3}\pi(a_0R)^3}{\pi R^2 \frac{L}{n}}$$

and the number of particles in the tube length $L$ therefore is

$$n = \frac{3}{4}\,\frac{c}{a_0^3}\,\frac{L}{R}$$

the expressions (16) and (17) take the form

$$\frac{\Delta p - \Delta p_0}{\Delta p_0} = \frac{3}{2}\,\frac{c}{a_0^3}\left\{\int_{s_0}^{s_1}\frac{ds}{1-a^4} - (s_1 - s_0)\right\} \qquad (18)$$

and

$$\frac{\mu^*}{\mu} = 1 + \frac{3}{2}\,\frac{c}{a_0^3}\left\{\int_{s_0}^{s_1}\frac{ds}{1-a^4} - (s_1 - s_0)\right\} \qquad (19)$$

respectively, where $c \leq \dfrac{\frac{4}{3}\pi a_0^3}{\pi 2 a_0} = \dfrac{2}{3}a_0^2$, since the particles must all be on the axis according to the assumptions of the present model.

By (6) the integral $I_0^1$ can be expressed by

$$I_0^1 = \int_{s_0}^{s_1}\frac{ds}{1-a^4} = N_{Re}\int_{a_1}^{a_0}\frac{1}{1-a^4}\;\frac{(1-a^2)\left(1-\frac{a^2}{3}\right)}{(1+a^2)^2}\,a\,da$$

or, with

$$J(a) = -\int\frac{1}{1-a^4}\;\frac{(1-a^2)\left(1-\frac{a^3}{3}\right)}{(1+a^2)^2}\,a\,da \qquad (20)$$

$$I_0^1 = N_{Re}\,[J(a_1) - J(a_0)]$$

$J(a)$ is plotted in Figure 4 as computed from (20).

The procedure for determining, for instance, the apparent viscosity of a suspension of spherical particles of radius $a_0$ and concentration $c$ in tube flow of Reynolds number $N_{Re}$ is then as follows:

1. The value of $s_0$ corresponding to $a_0$ is found in Figure 3.

2. If the distance between particles $L/nR = 4a_0^3/3c$ is larger than $(2(s^* - s_0)$, then

$$s_1 = s^* = 0.14\,N_{Re}.$$

Otherwise

$$s_1 = \frac{L}{2nR} - s_0 = \frac{2a_0^3}{3c} - s_0$$

3. The value of $a_1$ corresponding to $s_1$ is found in Figure 3.

4. The value of the integral $I_0^1$ is determined from Figure 4 as $N_{Re}\,[J(a_1) - J(a_0)]$.

5. $I_0^1$ and $s_1 - s_0$ are introduced in (19) to give the increase in apparent viscosity.

### Remark on the Error in the Model

From Schiller's Equation (12), and from (4) and (1) of the present paper, the friction force in the inlet per unit of tube length can be expressed by

$$F_1 = 8\pi\mu U_0 R\,\frac{\frac{q}{2}}{2 - \left(\frac{6}{q} - 2\right)^{1/2}} = 8\pi\mu U_0 R f_1(sN_{Re}^{-1})$$

and

$$F_2 = 8\pi\mu U_0 R\,\frac{\frac{q}{2}}{2 - \frac{2}{q}} = 8\pi\mu U_0 R f_2(sN_{Re}^{-1})$$

respectively. The ratios between the friction in the inlet and in the developed flow

$$f_1 = \frac{F_1}{8\pi\mu U_0 R}$$

and

$$f_2 = \frac{F_2}{8\pi\mu U_0 R}$$

respectively, are plotted in Figure 5. For small values of $sN_{Re}^{-1}$, where Schiller's model agrees well with observations, there is an appreciable difference between the two functions. However, since the truncated paraboloid velocity profile of the present model is exact at the particle, and thus the friction at this section is given by $f_2$, the true $f$ curve corresponding to the particle must run from the appropriate point on the $f_2$ curve, as shown by the tentative

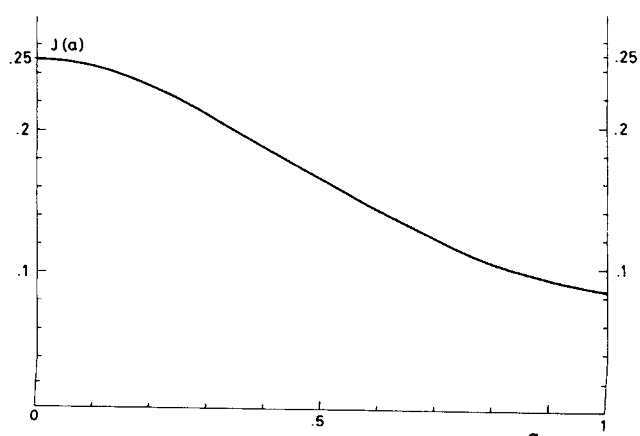


Fig. 4. Plot of $J(a)$ computed from Equation (20).

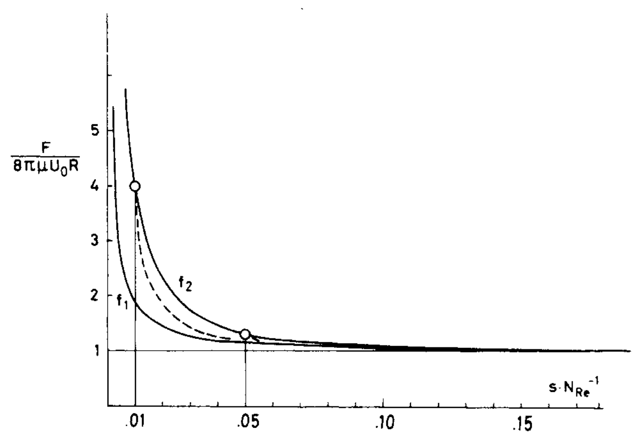


Fig. 5. Dimensionless wall friction in inlet flow according to Schiller's model, $f_1$, and according to the present model, $f_2$, with tentative curves for two cases of particle flow, namely, $a_0 = 0.8$ ($s_0 N_{Re}^{-1} = 0.01$) and $a_0 = 0.5$ ($s_0 N_{Re}^{-1} = 0.05$).

curves for $s_0 N_{Re}^{-1} = 0.01$, $a_0 \approx 0.8$, and for $s_0 N_{Re}^{-1} = 0.05$, $a_0 \approx 0.5$. The error in the present model, applied to particle flow, is therefore less than should be expected from its crudeness as an approximation to the inlet flow as indicated by Figure 2, especially since the additional resistance introduced by the particles will be proportional to the area under the $f$ curve in Figure 5, from $s_0 N_{Re}^{-1}$ to the appropriate value of $s_1 N_{Re}^{-1}$.

On the other hand, Figure 5 shows that the error may still be appreciable for particles that nearly fill the tube, being less the smaller the particle. The model is only valid for particles on the axis, however, and this probably puts a lower limit on the particle size, since Segré and Silberberg (7) have shown that small particles migrate from the axis.

## NOTATION

$a$ = dimensionless radius of core
$a_0$ = dimensionless radius of particle
$a_1$ = dimensionless radius at core intersection
$c$ = volume concentration of particles
$f_1$ = dimensionless friction force $\frac{F_1}{8\pi\mu U_0 R}$, according to Schiller's model
$f_2$ = dimensionless friction force $\frac{F_2}{8\pi\mu U_0 R}$, according to present model
$F_1$ = friction force per unit length in inlet flow, Schiller's model
$F_2$ = friction force per unit length in inlet flow, present model

$$I_0^* = \int_{s_0}^{s^*} \frac{1}{1 - a^4}\, ds$$

$$I_0^1 = \int_{s_0}^{s_1} \frac{1}{1 - a^4}\, ds$$

$$J(a) = -\int \frac{1}{1 - a^4} \frac{(1 - a^2)\left(1 - \frac{a^2}{3}\right)}{(1 + a^2)^2}\, a\, da$$

$L$ = length of tube
$n$ = number of particles in tube length $L$
$N_{Re}$ = tube Reynolds number $\frac{\rho U_0 R}{\mu}$
$p$ = pressure
$\Delta p$ = pressure drop over $L$ with particles in tube
$\Delta p_0$ = pressure drop over $L$ without particles
$\delta p$ = pressure drop associated with each particle
$q$ = dimensionless core velocity $U/U_0$
$r$ = dimensionless radial coordinate
$R$ = radius of tube
$s$ = dimensionless axial coordinate $x/R$
$s^*$ = dimensionless coordinate of core apex
$s_0$ = dimensionless coordinate of section where core and particle radii are equal
$s_1$ = dimensionless coordinate of core intersection
$u$ = axial velocity, function of $r$ and $s$
$U$ = axial velocity in the core, function of $s$
$U_0$ = axial flux velocity, constant
$x$ = axial coordinate

### Greek Letters

$\rho$ = fluid density
$\mu$ = dynamic viscosity
$\mu^*$ = apparent dynamic viscosity

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# Permeation Through Liquid Surfactant Membranes


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**A novel separation technique based on the selectivity of a liquid membrane composed of surfactants and water is described. The permeation mechanism is discussed in terms of surfactant concentration, surfactant structure and chain length, the nature of permeate, and the solubility of permeate in water.**


A novel separation technique using liquid surfactant membranes has been discovered (*1*) which is effective in separating hydrocarbons of different kinds, including those similar in their physical and chemical properties.

A liquid surfactant membrane, or abbreviated as liquid membrane, is a film composed of surfactants and their